IRJEMS International Research Journal of Economics and Management Studies
Published by Eternal Scientific Publications
ISSN: 2583 – 5238 / Volume 2 Issue 3 July 2023 / Pg. No: 187-196
Paper Id: IRJEMS-V2I3P124, Doi: 10.56472/25835238/IRJEMS-V2I3P124

Original Article

Measurement Methods of Risk Positions with Knight Uncertainty

[1]Zhaoyong Geng, [2]Yuquan Cui
[1,2]School of Mathematics, Shandong University, Jinan, China.

Received Date: 14 July 2023 Revised Date: 20 July 2023 Accepted Date: 25 July 2023 Published Date: 31 July 2023

Abstract: *This paper attempts to expand the scope of risk measurement to those risk positions with Knight uncertainty, which not only have "unknown unknown" but also have "known unknown". The first problem is the description of Knight's uncertainty. This paper summarizes and proposes a variety of definitions of Knight's uncertainty, describing the existence of Knight's uncertainty from subjective and objective perspectives. On this basis, several reasonable forms of risk measure that can simultaneously measure Knight uncertainty and risk are discussed. A method to measure the event's Knight uncertainty is proposed under the decision theory, and I have shown that this method of quantifying Knight uncertainty is consistent with Epstein's definition of unambiguous events and λ-systems. For the case of multiple prior probabilities, this paper uses the expectation of the maximum-minimum formula to define the measurement function. It takes the spectral risk measurement as an example to show that under certain conditions, the consistency condition of this measurement function is not different from the traditional spectral risk measurement function.*

Keywords*: Risk, Risk measure, Knight Uncertainty, Decision Theory, Coherence.*

I. INTRODUCTION

Risk measure is the basis and difficulty of the whole risk management process. Many of the disasters of risk management are due to the fact that the risk measurement model does not take into account some potential risks or overestimates the current risk value. The essence of risk measurement is the quantitative estimation of financial risk position, and mathematically it is functional from a position space (random variable space) to a real number set.

The research on risk measurement is to give specific mathematical forms of this function, from linear to sublinear, convexity, and homomonotone additivity. People are constantly exploring the properties of a suitable risk measurement. The famous Allais[1] paradox experiment shows that when the expected utility model is applied to practical decision-making, it will conclude inconsistent with rational intuition. People's perception of risk is not linear, and extreme losses bring more negative effects to people. In the 1990s, J.P. Morgan put forward the famous VaR measurement index. Because it is easy to calculate and understand, VaR is still in use today. Artzner[2] proposed the concept of consistent risk measurement. Later, four axioms of consistency became the most important properties of measuring a risk measurement. The concept of convex risk measurement proposed by Follmer[3] is an extension of consistent risk measurement, which covers a wider range. The ES measurement method proposed by Acerbi[4], as a substitute for VaR, solves the consistency problem. Later, Acerbi[5] proposed a more general spectral risk measure, including VaR and CVaR as special cases. The key concept of its spectral function can better reflect the risk aversion level of more people.

At the same time, the concept of distorting risk measurement appears in Wang's [6-7]'s research on insurance pricing. Julia[8] proved the necessary and sufficient conditions for the consistency of distortion risk measurement. Balbás[9] started with the distortion function of distortion risk measurement. They pointed out that even if the consistency has been met in some special cases, there may still be inconsistent results. It is worth mentioning that Wang's transformation has been proven to meet these properties of Balbás, but it cannot be proved that there will be no inconsistency. Peng[10] et al. set the variance of the normal distribution as the variable to reflect the uncertainty of the model and proposed the G-VaR measurement model. The operation of only considering the worst case in G-VaR is very similar to the idea of maximum and minimum expectations.

Knight[11] distinguishes risk from uncertainty. He called those variables that cannot be quantified uncertainty and those that can be quantified by statistical science real risk. In decision theory, risk is those acts whose decision results are unknown, but the decision-maker can fairly and accurately quantify the probability of each result. Knight regards uncertainty as the decision of those decision-makers who do not have enough information to obtain the probability of those results. Today, we call this real uncertainty Knight uncertainty. Risk and uncertainty are still regarded as the same concept.

Decision theory is a theory about consumer preferences or a theory that helps consumers choose between two positions. Since the preference theory does not require the accurate probability distribution of all decision results, it is a very desirable path to use the decision preference theory to describe the Knight uncertainty and make the final portfolio choice when the application scope of quantitative theoretical tools is limited.

When making decisions under the condition of uncertain probability, the first theory is the famous subjective probability theory proposed by Savage[12], which regards all decision options as having a certain probability distribution (subjective probability) and combines the utility function of Von Neumann[13] to use its mathematical expectation as the final effect to assist the decision. However, Savage's theory cannot explain the situation under the Ellsberg paradox, which will be detailed later. Schmeilder[14]proposed the concept of max-min expectation, which solved the problem of Ellsberg's paradox, and took Savage's theory as a special case.

Grant[15]et al. gave the definition of the basic growth of uncertainty, which is independent of the subjective probability model and any preference decision model with specific forms and can give the definition of uncertainty aversion according to this dual, which is similar to Epstein[16]. Paolo Ghirardato[17]et al. gave a definition of the relative form of uncertainty aversion by the preference theory. We will introduce it later and use it to qualitatively explain the impact of risk positions with Knight uncertainty on the portfolio.

The rest of the paper is organized as follows. Section 2 introduces the basic concepts of risk measurement theory and some existing descriptive tools for Knight uncertainty. Section 3 proposes two methods for describing Knight uncertainty within the decision theory framework and multiple prior probabilities. Section 4 concludes.

II. PRELIMINARIES

Let S be a sample space, and X be a random variable of loss or return under a given portfolio in a certain period.

Let X be the collection of all risk positions. Given a risk position X , here we uniformly define the risk position as loss, which means the smaller is, the better

Definition 2.1(Risk measure).

A risk measure is a functional that maps X to a positive real number field:

$$\rho : X \mapsto \mathbb{R}^{+}$$

Definition 2.2(Value at risk).

Given the quantile α , the Var with the confidence of X as α is defined as

$$VaR_X^{\alpha} = \inf\{x_{\alpha} \mid \int_{-\infty}^{x_{\alpha}} f(x)dx \geq \alpha\}$$

Where $f(x)$ is the probability density function of X , and α is its confidence.

It can be seen that when the density function is not constant zero in any interval or when the distribution function is strictly increased, the concept VaR is the same as that of the statistical quantile.

Artzner[2] proposed the concept of consistent risk measurement, which limits the risk measurement in the form of axioms to better reflect the investors' characteristics in the actual market.

Definition 2.3(Coherence).

If the risk measurement function ρ satisfies

(1) (Monotonicity) $\forall X, Y \in \mathrm{X}$, if $X \leq Y$, then $\rho(X) \leq \rho(Y)$

(2) (Subadditivity) $\forall X, Y \in \mathrm{X}$, $\rho(X+Y) \leq \rho(Y) + \rho(X)$

(3) (Translation invariance) $\forall X \in \mathrm{X}, a \in \mathbb{R}$, $\rho(X+a) \leq \rho(X) - a$

(4) (Positive homogeneity) $\forall X \in \mathrm{X}, \lambda \in \mathbb{R}^{+}$, $\rho(\lambda X) \leq \lambda \rho(X)$

Then ρ is called a coherent risk measure.

It can be proved that VaR it does not meet the sub-additivity, but $CVaR$ it does. However, we will not use $CVaR$ the below, so its definition will not be introduced here. Let's look directly at the definition of spectral risk measurement, a more generalized form of risk measure.

Definition 2.4 (Spectral risk measure).

Given a risk-averse function $\varphi:[0,1]\to[0,1]$, then the risk measure generated by φ is defined as

$$\rho_\varphi(X)=\int_0^1 VaR_\alpha(X)\varphi(\alpha)d\alpha$$

φ is called a risk spectrum. In turn, ρ_φ is called a spectral risk measure.

The reason why we introduce spectral risk measure here is that it has strong generality and can accommodate VaR and $CVaR$ as its special cases. The next theorem gives the sufficient and necessary condition of the spectral risk measure's coherence.

Theorem 2.1 Let ρ_φ be defined by

$$\rho_\varphi(X)=\int_0^1 VaR_\alpha(X)\varphi(\alpha)d\alpha$$

Then ρ_φ is coherent if and only if φ is an admissible risk spectrum. That is to say, φ satisfies

(1) φ is positive,

(2) φ is increasing,

(3) $\int_0^1 \varphi(\alpha)d\alpha=1$.

In the next section, we will show the consistency conditions of spectral risk measures for positions with knight uncertainty.

In the remainder of this section, we will introduce some concepts in decision theory, which is a good tool for qualitatively describing Knight's uncertainty.

First, we explain the existence of Knight uncertainty and the failure of Savage's subjective probability model through Ellsberg's three-colour urn experiment.

Example 2.1 There are 90 balls in an urn, 30 red ones and the rest of 60 either black or green. One ball will be drawn at random from the urn. Consider the following two positions based on drawn results:

$$X=\{100,R;0,B;0,G\}$$
$$Y=\{0,R;100,B;0,G\}$$

Where R,B,G denote the events corresponding to the chosen ball being red, black, or green.

Ellsberg's thinking experiment showed that people would prefer X to Y. This phenomenon can be explained by the deterministic effect. However, if we increase the reward for choosing the green ball in X and Y to 100, we will get two new positions:

$$X'=\{100,R;0,B;100,G\}$$
$$Y'=\{0,R;100,B;100,G\}$$

According to the principle of uncertainty aversion, people should tend to Y'. However, according to the subjective probability model of Savage, the opposite result will be obtained:

If the utility function $U:\mathrm{X}\to\mathbb{R}$ is in the expected form, then

$$U(X') = U(X) + u(100)\,p(G)$$
$$U(Y') = U(Y) + u(100)\,p(G)$$

Where $u:\mathbb{R}\to\mathbb{R}$ is a simple utility function only about the outcomes. So

$$U(X) > U(Y) \Leftrightarrow U(X') > U(Y')$$

Which contradicts the result of this experiment.

The key point of the paradox is that there are two kinds of events: unambiguous and ambiguous, but the SEU model cannot distinguish them since there is only one additive probability measure defined on all the events. We denote the unambiguous events' set as $\mathcal{A}$. Without causing confusion, the letters R, B, G will be used to refer to the number of corresponding colour balls. Besides, let S denote the state space and the corresponding σ algebra. Then in example 2.1,

$$S = \{R, B, G\}$$
$$\mathcal{A} = \{S, \varnothing, \{R\}, \{B, G\}\}$$

From the next example, we can see the set of unambiguous events is an algebra, but not a σ algebra.

Example 2.2. Suppose an urn with 100 balls, and a ball's colour may be black, red, green, or white. The sum of black and red is 50, and the sum of black and green balls is also 50. Like example 2.1, the following positions will be based on drawn results. We can see

$$\mathcal{A} = \{S, \varnothing, \{R, B\}, \{B, G\}, \{G, W\}, \{R, W\}\}$$

$\mathcal{A}$ is not an algebra since it is not intersection-closed:

$$\{B, G\} \in \mathcal{A}, \{B, R\} \in \mathcal{A} \nRightarrow B \in \mathcal{A}$$

So, we need a new algebraic system to reflect its characteristics:

Definition 2.5. (λ-systems).

A non-empty event set $\mathcal{A}$ is a λ-system if

(1) $S \in \mathcal{A}$

(2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

(3) $A_1, \cdots, A_n \in \mathcal{A}, A_1 \cup \cdots \cup A_n = \varnothing \Rightarrow A_1 \cup \cdots \cup A_n \in \mathcal{A}$

The biggest difference between σ algebra and λ-system is the closeness of the intersection operation. Condition (1) (2) (3) is also nature. From a certain point of view, $\mathcal{A}$ is a set of known information that can be used as an investor. So the definition of $\mathcal{A}$ can also be used as a description of Knight's uncertainty.

Based on λ-systems, there are several definitions of Knight's uncertainty.

First, we need to make some axiomatic restrictions on the ordinal relationship $\succeq$ to ensure that it conforms to the psychological preferences of a normal rational investor:

Axiom 2.1 (Monotonicity)

$$\forall X, Y \in \mathcal{X}, X(s) \geq Y(s), \forall s \in S \Rightarrow X \succeq Y$$

Axiom 2.2 (Certainty Equivalent)

$$\forall X \in \mathrm{X}, \exists x \in \mathbb{R}, X \sim x, \text{ i.e. } X \succeq x \text{ and } x \succeq X$$

Definition 2.6 Given a preference $\succeq$ and an event $A \subseteq \Sigma$, for any two events $B, C \subseteq \Sigma$ that do not intersect and do not intersect with A. Set the preference relationship between the two existing positions is

$$\{b, B; 0, B^c\} \succeq \{c, C; 0, C^c\}$$

Where $b, c \in \mathbb{R}^+$.

Then $A \in \mathrm{A}$ if

$$\{b, B; 0, (B \cup A)^c; a, A\} \succeq \{c, C; 0, (C \cup A)^c; a, A\}$$
$$\Leftrightarrow \{b, B; 0, B^c\} \succeq \{c, C; 0, C^c\}$$

For any $a \in \mathbb{R}$.

After adding the loss/gain corresponding to event A, the preference relationship between the two positions remains unchanged. This is the separability of unambiguous events as if the parts related to A on both sides of the equation were omitted. In section 3, my proposed Knight uncertainty measure $\beta(\cdot)$ can also reflect this characteristic of unambiguous events.

III. DESCRIPTION AND MEASUREMENT OF KNIGHT UNCERTAINTY

A) Multiple Prior Expected Utility

In the case of considering knight uncertainty, the distribution function of risk position X is expressed in the form of a family rather than one, and it is recorded as $\{F_\theta\}_{\theta \in \Theta}$.

Each state of the world corresponds to a distribution function, so for one of the distribution functions F_θ, its corresponding risk measure value is recorded as

$$\rho_\varphi^\theta(X) = \int_0^1 \varphi(\alpha) VaR_\alpha^\theta(X) d\alpha$$

According to the idea of max-min expected utility, the spectral risk measure form considering Knight uncertainty should be as follows:

$$\hat{\rho}_\varphi(X) = \sup_{\theta \in \Theta} \int_0^1 \varphi(\alpha) VaR_\alpha^\theta(X) d\alpha \quad \text{--} \quad (1)$$

The following theorem shows that the consistency condition of spectral risk measure with knight uncertainty defined by the above formula is not different from the original, i.e., the traditional one.

Theorem 3.1 If $\{F_\theta\}_{\theta \in \Theta}$ is a weakly compact set, then the risk measure defined by (3-1) is coherent if and only if φ is an admissible risk spectrum.

Proof. For arbitrary $w \in \Theta$. The following formula is obviously true

$$\int_0^1 \varphi(\alpha) VaR_\alpha^w(X) d\alpha \le \sup_{\theta \in \Theta} \int_0^1 \varphi(\alpha) VaR_\alpha^\theta(X) d\alpha \le \int_0^1 \varphi(\alpha) \sup_{\theta \in \Theta} VaR_\alpha^\theta(X) d\alpha \quad \text{-----------------------} \quad (2)$$

Let

$$\hat{F}(x) = \sup_{\theta \in \Theta} F_\theta(x)$$

It is easy to prove that $\hat{F}(x)$ is still a distribution function, satisfying the characteristics of right continuity, increasing, and range of [0,1].

Set

$$VaR_{\alpha}^{*}(X)=\hat{F}^{-1}(\alpha)$$

i.e., the VaR value under the condition that the distribution function is $\hat{F}(x)$

As $\{F_{\theta}\}_{\theta\in\Theta}$ is compact, we have

There exists $\hat{\theta}\in\Theta$, s.t.

$$VaR_{\alpha}^{*}(X)=\sup_{\theta\in\Theta}VaR_{\alpha}^{\theta}(X)=VaR_{\alpha}^{\hat{\theta}}(X)$$

In the inequality (3-2), we can obtain the equation following by set $w=\hat{\theta}$:

$$\int_0^1\varphi(\alpha)VaR_{\alpha}^{\hat{\theta}}(X)d\alpha=\sup_{\theta\in\Theta}\int_0^1\varphi(\alpha)VaR_{\alpha}^{\theta}(X)d\alpha$$

Then we can see $\hat{\rho}_{\varphi}$ is still in the form of a spectrum, so its coherent conditions are the same as Theorem 2.1.

B) A Measurement of Knight Uncertainty Under Decision Theory

As mentioned in section 2, the sources of Knight's uncertainty in decision theory are those events outside the unambiguous event set $\mathcal{A}$ .So a natural idea of investors is to use the event probability information inside $\mathcal{A}$ to approximate probabilities of those events outside $\mathcal{A}$. Let's first introduce the concept of upper and lower measures.

Definition 3.1: Given an unambiguous event set $\mathcal{A}$ and an additive probability measure p defined on it. Then for any event $B\in\Sigma$, the upper measure of it is defined as

$$p^{*}(B)=\{\inf p(A)\mid A\in\mathcal{A},A\supseteq B\}$$

Similarly, the lower measure is

$$p_{*}(B)=\{\sup p(A)\mid A\in\mathcal{A},A\subseteq B\}$$

There is an intuitive explanation for the concept of upper and lower measures： When an investor uses the upper measure to represent his expected probability of an event, this value represents the probability of the event in the best case, and the investor's mindset is optimistic. Conversely, when an investor uses the lower measure to represent his expected probability of an event, this value indicates the probability of the event in the worst-case scenario, and the investor's mentality is pessimistic.

Using the concepts of upper and lower measures, we can obtain a measure of the degree of Knight's uncertainty about events.

Definition 3.2 Given an unambiguous event set $\mathcal{A}$ and an additive probability measure p defined on it. Then for every event $B\in\Sigma$, its knight uncertainty value is defined as

$$\beta(B)=p^{*}(B)-p_{*}(B) \qquad \text{---------------------------------------(3)}$$

Where $\beta:\Sigma\to\mathbb{R}^{+}$ is a knight uncertainty measure

This idea is intuitive. Firstly, for an event B with Knight uncertainty, the values of its upper and lower measures represent the upper and lower limits of its possible occurrence probability, respectively. The smaller the difference between the upper and lower limits, the more certain the range of its probability value is, and the better the "approximation" of the information that investors already have to it.

We first apply it to Ellsberg's three-colour urn experiment.
In this experiment,

$$\mathrm{A} = \{S, \varnothing, \{R\}, \{B, G\}\}$$

We choose an ambiguous event $\{B\}$ whose probability is uncertain and then

$$p^*(B) = p(\{B, G\}) = \frac{2}{3}$$

$$p_*(B) = p(\varnothing) = 0$$

By (3), we have

$$\beta(B) = p^*(B) - p_*(B) = \frac{2}{3}$$

Let's see another event $\{R, B\}$. Similarly,

$$p^*(\{R, B\}) = p(S) = 1$$

$$p_*(\{R, B\}) = p(R) = \frac{1}{3}$$

$$\beta(\{R, B\}) = p^*(\{R, B\}) - p_*(\{R, B\}) = \frac{2}{3}$$

We can find that in the three-colour urn experiment, the Knight uncertainty metric for the event $\{B\}$ and $\{R, B\}$ is the same. In fact, this fact validates the rationality of definition 3.2, which is consistent with definition 2.6. Both utilize the separability of deterministic events. That is the property (3) in the definition of λ -systems.

Besides, the property (2) should also be noticed. The next proposition illustrates that definition 3.2 satisfies this property.

Proposition 3.1: The knight uncertainty measure $\beta(\cdot)$ defined in definition 3.2 satisfies the complementary symmetry condition. i.e.

$$\beta(B) = \beta(B^c), \forall B \in \Sigma \qquad \text{------------------------------------(4)}$$

Proof. Set events $A, B \in \Sigma$ which satisfy

$$A \subseteq B, A \notin \mathrm{A}, B \in \mathrm{A}$$

As the following Venn shows,

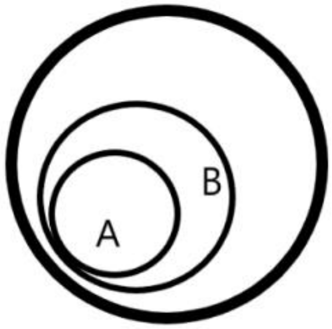

Fig.1 The maximum circle represents the complete set of states S.

By definition 3.1,

$$p^*(A) = p(B)$$

From the Venn, it is easy to see

$$p_*(A^c) = p(B^c) = 1 - p(B) = 1 - p^*(A)$$

From the symmetry, it can also be obtained that

$$p_*(A) = 1 - p^*(A^c)$$

So we have

$$p^*(A) - p_*(A) = (1 - p_*(A^c)) - (1 - p^*(A^c)) = p^*(A^c) - p_*(A^c)$$

That is,

$$\beta(A) = \beta(A^c), \forall A \notin \mathcal{A}$$

As for those unambiguous events, (2) is obvious, as

$$\beta(B) = 0, \forall B \in \mathrm{A} \quad ,$$

In conjunction with the von Neumann utility function, we can extend the Knight uncertainty metric in definition 3.2 to the definition domain of risk positions. For example, the simplest and appropriate form of a Knight uncertainty measure function is the product of the Knight uncertainty measure of an event and the degree of dispersion of the corresponding result of the event.

Definition 3.3 $\forall X = \{x_1, E_1; x_2, E_2; \cdots x_n, E_n\} \in \mathrm{X}$ defines its Knight uncertainty measure as

$$\beta(X) = \sum_{j=1}^{n} |u(x_{j+1}) - u(x_j)| (p^*(\bigcup_{i=j}^{n} E_i) - p_*(\bigcup_{i=j}^{n} E_i))$$

As a representation of Knight's uncertainty, capacity also reflects investor's Knight uncertainty aversion. The relationship between the capacity and $\beta(\cdot)$ is worth exploring.

Remark 3.1 By any Knight uncertainty measure $\beta(\cdot)$ that satisfies the following conditions, a capacity can be obtained:

(1)

$$A, B \in \Sigma, A, B \notin \mathrm{A}, A \subseteq B$$
$$\Rightarrow \beta(x, A; 0, A^c) \geq \beta(x, B; 0, B^c), \forall x \in \square^+$$

(2)

$$\forall A \in \mathcal{A}, a, b \in \mathbb{R}, \beta(a, A; b, A^c) = 0$$

(3) There exists an additive probability measure $p(\cdot)$ defined in the unambiguous events set A

(4) The utility function $u(\cdot)$ satisfies $u(0) = 0$ and the monotonicity

Proof. Let

$$\beta'(x, A; 0, A^c) = \begin{cases} \beta(x, A; 0, A^c), & A \notin \mathcal{A} \\ u(x) p(A), & A \in \mathcal{A} \end{cases}$$

$$\forall A \in \Sigma, x \in \mathbb{R}^+$$

From this, we can obtain a standard capacity:

$$v_x(A) = \frac{\beta'(x, A; 0, A^c)}{\beta'(x, S)}$$

IV. CONCLUSION

In this paper, I propose a method for measuring Knight uncertainty under decision theory and prove its compatibility with some existing methods for describing Knight uncertainty. The theorem on the consistency condition of spectral risk measures with Knight uncertainty also demonstrates the rationality of the maximum-minimum expectation.

However, under the decision theory framework, using upper and lower measures to measure the Knight uncertainty of events still does not fully utilize known probability information. For example, in a four-colour urn experiment, if the sum of the numbers of two colour balls is known to be different, the minimum number of two types of balls can be obtained. But the design of the upper and lower measures does not utilize this information, which may lead to inconsistent results.

For situations where multiple prior probabilities are used to represent Knight's uncertainty, taking the upper bound of the corresponding risk measure is consistent with traders' cautious practice in the current stress testing market. However, asset pricing will lose most of the information about probability distribution families. More complex forms of measurement of Knight's uncertainty await discovery.

Conflicts of Interest
The authors declare that there is no conflict of interest concerning the publishing of this paper.

Funding Statement
This paper has no funding support.

V. REFERENCES

[1] Allais M. Allais paradox[M]. Palgrave Macmillan UK, 1990.
[2] Artzner P, Delbaen F, Eber J M, et al. Coherent measures of risk[J]. *Mathematical finance*, 1999, 9(3): 203-228.
[3] Föllmer H, Schied A. Convex measures of risk and trading constraints[J]. *Finance and stochastics*, 2002, 6: 429-447.
[4] Carlo Acerbi, Dirk Tasche. Expected Shortfall: A Natural Coherent Alternative to Value at Risk[J]. *Economic Notes*, 2002, 31: 379-388.
[5] Acerbi C. Spectral measures of risk: A coherent representation of subjective risk aversion[J]. *Journal of Banking & Finance*, 2002, 26(7): 1505-1518.
[6] Wang S. Premium calculation by transforming the layer premium density[J]. ASTIN Bulletin: *The Journal of the IAA*, 1996, 26(1): 71-92.
[7] Wang S. A class of distortion operators for pricing financial and insurance risks[J]. *Journal of risk and insurance*, 2000: 15-36.
[8] Wirch J L, Hardy M R. Distortion risk measures: Coherence and stochastic dominance[C]//International congress on insurance: *Mathematics and economics*. 2001: 15-17.
[9] Alejandro Balbás,José Garrido,Silvia Mayoral. Properties of Distortion Risk Measures[J]. Methodology and Computing in Applied Probability, 2008, 11: 1-2.
[10] Shige Peng, Shuzhen Yang, Jianfeng Yao. Improving Value-at-Risk Prediction Under Model Uncertainty[J]. *Journal of Financial Econometrics*, 2020, 21: 1-2.
[11] Knight F H. Risk, uncertainty and profit[M]. Houghton Mifflin, 1921.
[12] Ellsberg D. Risk, ambiguity, and the Savage axioms[J]. *The quarterly journal of economics*, 1961, 75(4): 643-669.

[13] Rubinstein A, Kuhn H W, Morgenstern O, et al. Theory of Games and Economic Behavior: 60th Anniversary Commemorative Edition[M]. Princeton university press, 2007.
[14] Schmeidler D. Subjective probability and expected utility without additivity[J]. Econometrica: *Journal of the Econometric Society*, 1989: 571-587.
[15] Simon Grant, John Quiggin. Increasing uncertainty: a definition[J]. *Mathematical Social Sciences*, 2005, 49: 117-141.
[16] Epstein L G. A definition of uncertainty aversion[M]//Uncertainty in economic theory. Routledge, 2004: 187-224.
[17] Ghirardato P, Marinacci M. Ambiguity made precise: A comparative foundation[J]. *Journal of Economic Theory*, 2002, 102(2): 251-289.